

Lecture 1 (Feb 1, 2016)

Goal: study the stability of systems that are governed by finite coupled ODEs of the form:

Non linear state systems

$$\dot{x}_1 = f_1(t, x_1, \dots, x_n) \quad x_1(0) = x_1^0$$

$$\vdots$$

$$\dot{x}_n = f_n(t, x_1, \dots, x_n) \quad x_n(0) = x_n^0$$

$$\vdots$$

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$: state of the system

state variables x_1, \dots, x_n are physical quantities which when specified completely determine the evolution of the system.

- \dot{x}_i : derivatives of x_i with respect to time variable t .

autonomous (time-invariant)

A system is autonomous if f does not depend explicitly on time t :

$$\dot{x} = f(x), \quad x(0) = x_0$$

otherwise it is non-autonomous (time-varying).

Non linear Control system:

If the system dynamics can be influenced by "control inputs" $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, we describe the control system as follows:

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, \dots, x_n, u_1, \dots, u_p) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n, u_1, \dots, u_p)\end{aligned} \Rightarrow \dot{x} = \tilde{f}(t, x, u)$$

A feedback control law $u = u(t, x)$ yields a closed loop (or controlled) system $\dot{x} = \tilde{f}(t, x, u(t, x)) = \tilde{f}(t, x)$.

Non linear Control system with output:

Sometimes certain "output variables" $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ are of interest. These might be measured variables or special variables that one care about.

The state space model with output equation is

$$\begin{cases} \dot{x} = f(t, x, u) \\ y = h(t, x, u) \end{cases}$$

Equilibria

- A point $x = x^*$ is an equilibrium of $\dot{x} = f(t, x)$ if $f(t, x^*) = 0, \forall t$.
- x^* is an isolated equilibrium if \exists some $\delta > 0$ s.t. there is no other equilibrium in the ball $B(x^*, \delta) = \{x \mid \|x - x^*\| < \delta\}$
- Assume $f(t, x)$ is C^1 (continuously differentiable on x).

The linearization about an equilibrium x^* is given by

$$\dot{z} = A(t) z$$

where $z = x - x^*$ and $A(t)$ is the $n \times n$ matrix

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=x^*}$$

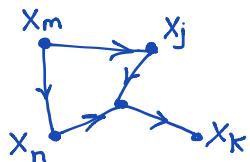
$$\text{with } A_{ij} = \frac{\partial f_i}{\partial x_j}.$$

For an autonomous system $A(t) = A$ is constant.

Prop. If $A(t) = A$ is nonsingular, then x^* is an isolated equi.

Example 1. Multi-agent system dynamics

consider N agents x_1, \dots, x_N with $x_k \in \mathbb{R}$ which are connected through an arbitrary graph.



The following set of odes describe the dynamics of N interconnected agents:

$$\dot{x}_i = \sum_{k \rightarrow i} a_{ik} (x_k - x_i) \quad (*)$$

$$\dot{x}_N = \sum_{k \rightarrow N} a_{Nk} (x_k - x_N)$$

where a_{kj} 's are the weights of connections and

$a_{kj} > 0$ if k can sense relative state of j

$a_{kj} = 0$ otherwise

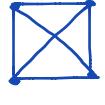
One can write (*) as follows:

$$\dot{x} = -Lx$$

where $x = (x_1, \dots, x_N)^T$ and L is the Laplacian matrix of the graph and is defined as follows:

$$L = (l_{kj}) , \quad l_{kj} = \begin{cases} -\sum_{i=1, i \neq j}^N a_{ki} & k=j \\ a_{kj} & k \neq j \end{cases}$$

example 1) complete graph with unit weight and $N=4$ nodes


$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

example 2) cycle graph with unit weight and $N=4$ nodes


$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Note that

$$L \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \Rightarrow 0 \text{ is always an eigenvalue with eigenvector } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If graph is connected, nullspace (L) has rank 1.

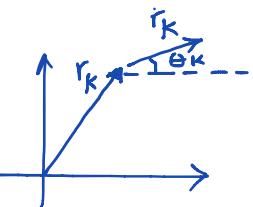
Then $x^* = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an equilibrium of $(*)$ for any $\alpha \in \mathbb{R}$.

$(*)$ is called consensus dynamics.

One can show that all other eigenvalues of $-L$ have negative real part. For any $x(0)$, $x(t) \rightarrow \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for some $\alpha \in \mathbb{R}$.

example 2.

consider N agents (particles) that move in the plane at constant (unit) speed. For $k=1, \dots, N$

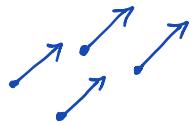

$$\dot{r}_k = \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}$$
$$\dot{\theta}_k = u_k = \text{steering control}$$

Consider steering law and θ_k dynamics only

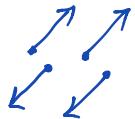
$$\dot{\theta}_k = \sum_{j=1}^N a_{kj} \sin(\theta_j - \theta_k) = u_k$$

Equilibria correspond to $\sum_{j=1}^N a_{kj} \sin(\theta_j - \theta_k) = 0, k=1, \dots, N$

$$\textcircled{1} \quad \theta_k = \bar{\theta}, \quad \forall k = 1, \dots, N$$

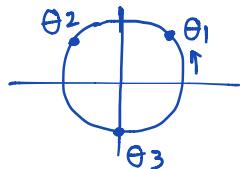


$$\textcircled{2} \quad \theta_k = \bar{\theta} \text{ or } \bar{\theta} + \pi$$



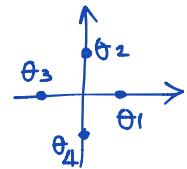
Example : complete graph with unit weight , $N=3$

$$\begin{aligned}\dot{\theta}_1 &= \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1) \\ &= \sin \frac{2\pi}{3} + \sin \frac{-2\pi}{3} = 0\end{aligned}$$



Example : complete graph with unit weight , $N=4$

$$\begin{aligned}\dot{\theta}_1 &= \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1) + \sin(\theta_4 - \theta_1) \\ &= \sin \frac{\pi}{2} + \sin \pi + \sin \frac{3\pi}{2} = 0\end{aligned}$$



predator-prey (Lotka-Volterra) model

independently Alfred Lotka & Vito Volterra , in 1920's

Assumptions :

- 1) Prey populations grows exponentially when predator is absent.
- 2) Predators will starve in the absence of prey (rather than switch to another prey)
- 3) Predators can consume infinite quantities of prey.
- 4) There is no environmental complexity (both prey & predators move randomly through a homogeneous environment)

Let $P = \# \text{ of predators}$

$N = \# \text{ of preys}$

$r = \text{growth rate of preys}$

$\alpha = \text{search efficiency / attack rate}$

$c = \text{predator's efficiency at turning food into offspring (conversion efficiency)}$

$q = \text{Predator mortality rate}$

$$\text{Model : } \frac{dP}{dt} = c\alpha PN - qP$$

$$\frac{dN}{dt} = rN - \alpha PN$$

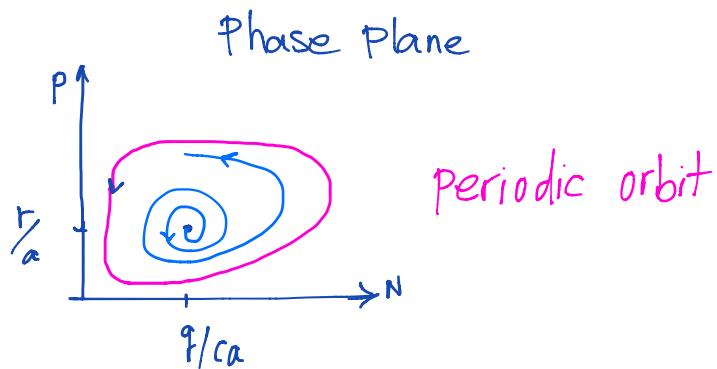
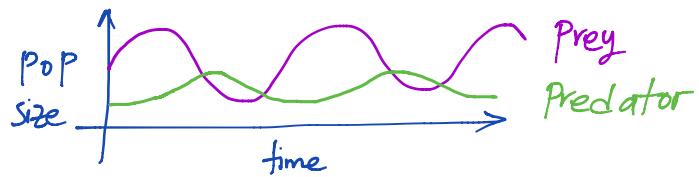
$$\text{Equilibrium points : } \frac{dP}{dt} = 0 \leftrightarrow P(c\alpha N - q) = 0$$

$$\frac{dN}{dt} = 0 \leftrightarrow N(r - \alpha P) = 0$$

1) $P = N = 0$ (not interesting)

2) $P = \frac{r}{a}$ and $N = \frac{q}{ca}$

when integrate equations of motion get oscillatory behavior!



Finite escape time: The state of an unstable linear system goes to infinity only when $t \rightarrow +\infty$.

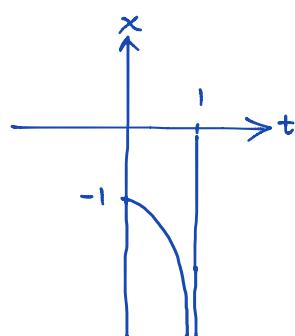
Consider the following non-linear system

$$\dot{x} = -x^2, \quad x(0) = -1.$$

Solve the differential equations $x(t) = \frac{1}{t-1}$.

As $t \rightarrow +1^-$, $x(t) \rightarrow -\infty$

(finite escape time)



How different is a non linear system from a linear system?

- A linear system gives only information around a fixed point, so it gives local results. A non-linear system gives global results.

- Nonlinear phenomena

- multiple isolated equilibrium

In a linear system $\dot{x} = Ax$, if

A : non-singular $\Rightarrow x=0$ is the only equilibrium.

A : singular \Rightarrow There may exist a sequence of equilibrium.

In a non-linear system $\dot{x} = f(t, x)$,

there may exist multiple isolated equilibrium.

- limit cycles
 - finite escape time